

Numerical ranges of $C_0(N)$ contractions

Chafiq Benhida, Pamela Gorkin and Dan Timotin

Mathematics Subject Classification (2010). 47A12, 47A20.

Keywords. Contraction, unitary dilation, numerical range.

Abstract. A conjecture of Halmos proved by Choi and Li states that the closure of the numerical range of a contraction on a Hilbert space is the intersection of the closure of the numerical ranges of all its unitary dilations. We show that for $C_0(N)$ contractions one can restrict the intersection to a smaller family of dilations. This generalizes a finite dimensional result of Gau and Wu.

1. Introduction

Suppose $\mathcal{H}, \mathcal{H}'$ are separable Hilbert spaces; we will denote by $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ the space of bounded linear operators $T : \mathcal{H} \rightarrow \mathcal{H}'$ and $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. The numerical range of an operator $T \in \mathcal{L}(\mathcal{H})$ is the set

$$W(T) := \{\langle Tx, x \rangle : \|x\| = 1\}.$$

Much is known about this set; for example, it is convex, in the finite-dimensional case it is compact, and if T is normal, the closure of $W(T)$ is the convex hull of the spectrum of T . In general, however, the numerical range is difficult to compute. In this paper, we study new ways of obtaining the numerical range of a contraction T from the numerical ranges of certain unitary dilations of T .

If there is a Hilbert space \mathcal{K} containing \mathcal{H} and an operator $\tilde{T} \in \mathcal{L}(\mathcal{K})$ such that $T = P_{\mathcal{H}}\tilde{T}|_{\mathcal{H}}$, where $P_{\mathcal{H}}$ denotes the orthogonal projection onto \mathcal{H} , the operator T is said to *dilate* to the operator \tilde{T} . (We note that we are considering the so-called *weak dilations* here, and not power dilations treated in Sz.-Nagy dilation theory.) The operator \tilde{T} is said to be a *dilation of T* ; more precisely, if $\dim(\mathcal{K} \ominus \mathcal{H}) = k$, then \tilde{T} is called a *k-dilation*.

We will be interested in unitary dilations. A result of Halmos [14, Problem 222(a)] shows that every contraction T has unitary dilations. It is easy to see that

$$\overline{W(T)} \subseteq \cap \{\overline{W(U)} : U \text{ is a unitary dilation of } T\}.$$

Choi and Li showed that, in fact,

$$\overline{W(T)} = \cap \{ \overline{W(U)} : U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } T \},$$

answering a question raised by Halmos (see, for example, [13]). We note that in the case that \mathcal{H} is n -dimensional, these unitary dilations are n -dilations; that is, the dilations are of size $2n \times 2n$.

Before Choi and Li's work was completed, Gau and Wu [9] studied the so-called compressions of the shift on finite-dimensional spaces and their numerical ranges. If SS_n is the class of all completely nonunitary contractions T (that is, $\|T\| \leq 1$ and T has no eigenvalue of modulus one) on an n -dimensional space with $\text{rank}(I - T^*T) = 1$, Gau and Wu [9, Corollary 2.8] showed that, in fact, if $T \in SS_n$, then

$$W(T) = \bigcap \{ W(U) : U \text{ is an } (n+1)\text{-dimensional unitary dilation of } T \}.$$

(There is no need to take the closure in the case of finite-dimensional spaces.) Thus, the unitary dilations may be chosen to be 1-dilations when $\text{rank}(I - T^*T) = 1$. An extension of this result can be found in [8]: namely, if T is an $n \times n$ contraction with $\text{rank}(I - T^*T) = k$, then

$$W(T) = \bigcap \{ W(U) : U \in M_{n+k} \text{ is a unitary } k\text{-dilation of } T \}. \quad (1.1)$$

It is easy to see that if $\text{rank}(I - T^*T) = k$, then T has no unitary ℓ -dilations for $\ell < k$, which explains why Gau, Li and Wu refer to (1.1) in [8] as the most “economical” solution to the Halmos problem. We also refer the reader to the papers [10], [11], [12], and [20] for work related to this discussion. These authors, as well as others, (in particular, [16], [17], and [5]) have studied this problem from a geometric point of view.

The analogue of SS_n on a space of infinite dimension is the class of contractions with $\text{rank}(I - T^*T) = \text{rank}(I - TT^*) = 1$ for which T^n and T^{*n} tend strongly to 0. It is well known (see, for instance, [19]) that such a T is unitarily equivalent to some *model operator* S_θ defined as follows: Suppose S is the unilateral shift on H^2 . For θ an inner function on the unit disc \mathbb{D} , define $K_\theta = H^2 \ominus \theta H^2$ and $S_\theta = P_{K_\theta} S|_{K_\theta}$. (The operator S_θ is often called a *compression* of the shift.) Noting that when $\theta(0) = 0$ all unitary 1-dilations of S_θ are equivalent to rank-1 perturbations of $S_{z\theta}$, the authors of [3] show that when $\theta = B$ is a Blaschke product we have

$$\overline{W(S_B)} = \bigcap \{ \overline{W(U)} : U \text{ a rank-1 perturbation of } S_{zB} \}.$$

Our goal in this paper is to extend these results to operator-valued inner functions. After two preliminary sections, the main results appear in Section 4, where we show that the closure of the numerical range of S_Θ , where Θ is an inner function in $H^2(\mathbb{C}^N)$, is the intersection of the closures of the numerical ranges of an appropriate family of unitary dilations of S_Θ (see Corollary 4.7). In Theorem 4.8 this result is extended to a larger class of contractions, called $C_0(N)$ (see Definition 2.3). In Section 5, we describe the spectrum of the unitary dilations, obtaining a generalization of the scalar

case. We conclude the paper with a brief discussion of a conjecture about the numerical ranges of contractions with finite defect index.

2. Preliminaries

2.1. Matrix-valued analytic functions

The basic reference that we will use for matrix-valued analytic functions (or, equivalently, functions with values in $\mathcal{L}(\mathbb{C}^N)$) is [15]; our definitions are simpler since we will consider only bounded (in the operator norm) analytic functions $F : \mathbb{D} \rightarrow \mathcal{M}_N$ (the set of $N \times N$ matrices). These share certain factorization properties similar to those of scalar analytic functions.

A bounded analytic matrix-valued function $F : \mathbb{D} \rightarrow \mathcal{M}_N$ is called *outer* if $\det F(z)$ is outer, and *inner* if the boundary values (which can be defined as radial limits almost everywhere) are isometries for almost all $e^{it} \in \mathbb{T}$.

It is known [15, Theorem 5.4] that any analytic bounded F can be factorized as

$$F = \Theta E \quad (2.1)$$

where Θ is inner and E is outer, and, if $F = \hat{\Theta} \hat{E}$, then $\hat{\Theta} = \Theta V$, $\hat{E} = V^* E$ for some constant unitary V .

The inner function appearing in (2.1) can be further factorized in two parts. Recall that a Blaschke–Potapov factor $b(P, \lambda)(z)$ determined by a point $\lambda \in \mathbb{D}$ and an orthogonal projection P on \mathbb{C}^N is an inner function given by the formulas

$$b(P, \lambda)(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda}z} P + (I - P) \text{ for } \lambda \neq 0, \quad b(P, 0) = zP + (I - P).$$

A finite Blaschke–Potapov product is a product

$$B_n(z) = b(P_1, \lambda_1)(z) \cdots b(P_n, \lambda_n)(z),$$

for some λ_j, P_j , $j = 1, \dots, n$. If (λ_j) is a Blaschke sequence in \mathbb{D} (that is, $\sum_j (1 - |\lambda_j|) < \infty$), while P_j is an arbitrary sequence of projections on \mathbb{C}^N , then the sequence $B_n(z)$ converges at each point $z \in \mathbb{D}$ to $B(z)$, where B is

an inner function denoted by $\prod_j b(P_j, \lambda_j)$. A function that can be written as $B(z)V$, where V is a constant unitary, is called an (infinite) Blaschke–Potapov product. The convergence is uniform on all compact subsets of \mathbb{D} . (Note that such a function is sometimes called a *left* Blaschke–Potapov product; we will not have the occasion to use right Blaschke–Potapov products.) Finally, an inner function Θ is called *singular* if $\det \Theta(z) \neq 0$ for all $z \in \mathbb{D}$.

With these definitions, Theorem 4.1 in [15] states that any inner function Θ decomposes as $\Theta = BS$, where B is a (finite or infinite) Blaschke–Potapov product and S is singular. As in the case of inner–outer factorization, the decomposition is unique up to a unitary constant; more precisely, if we also have $\Theta = \hat{B}\hat{S}$ with \hat{B} a Blaschke–Potapov product and \hat{S} singular, then $\hat{B} = BV$ and $\hat{S} = V^*S$ for some constant unitary V .

The next lemma is a Frostman-type theorem that follows from [15].

Lemma 2.1. *Every inner function Θ in $H^2(\mathbb{C}^N)$ is a uniform limit of infinite Blaschke–Potapov products.*

Proof. For $\lambda \in \mathbb{D}$, $(\Theta - \lambda I)(I - \bar{\lambda}\Theta)^{-1}$ is inner and $I - \lambda\Theta$ is outer; thus

$$\Theta - \lambda I = ((\Theta - \lambda I)(I - \bar{\lambda}\Theta)^{-1})(I - \bar{\lambda}\Theta)$$

is the inner–outer factorization of $\Theta - \lambda I$. But Corollary 6.1 from [15] says that for a dense set of $\lambda \in \mathbb{D}$ the inner factor of $\Theta - \lambda I$ is a Blaschke–Potapov product. If we take a sequence $\lambda_n \rightarrow 0$ with this property and we denote the corresponding Blaschke–Potapov product by $B^{(n)}$, then

$$B^{(n)} = (\Theta - \lambda_n I)(I - \bar{\lambda}_n \Theta)^{-1},$$

whence

$$\Theta = \lambda_n I + B^{(n)}(I - \bar{\lambda}_n \Theta) = \lim_{n \rightarrow \infty} B^{(n)}. \quad \square$$

2.2. Model spaces

Let $\mathcal{E}, \mathcal{E}_*$ be Hilbert spaces. Suppose we are given an operator-valued inner function $\Theta(z) : \mathcal{E} \rightarrow \mathcal{E}_*$. The *model space* associated to it is

$$K_\Theta := H^2(\mathcal{E}_*) \ominus \Theta H^2(\mathcal{E}),$$

The operator $\mathbf{T}_\Theta : H^2(\mathcal{E}) \rightarrow H^2(\mathcal{E}_*)$ defined by $\mathbf{T}_\Theta f = \Theta f$ is an isometry, and we have

$$P_{K_\Theta} = I - \mathbf{T}_\Theta \mathbf{T}_\Theta^*. \quad (2.2)$$

In particular, $\Theta(z) = zI_{\mathcal{E}_*} : \mathcal{E}_* \rightarrow \mathcal{E}_*$ is inner; we will denote the corresponding \mathbf{T}_Θ simply by \mathbf{T}_z . The *model operator* S_Θ is the compression of \mathbf{T}_z to K_Θ ; that is, $S_\Theta = P_{K_\Theta} \mathbf{T}_z P_{K_\Theta}|_{K_\Theta}$.

An inner function Θ is called *pure* if it has no constant unitary direct summand; this is equivalent to assuming $\|\Theta(0)x\| < \|x\|$ for all $x \neq 0$. A general inner function is the direct sum of a pure inner function and a unitary constant; from the point of view of model spaces and operators we may consider only pure inner functions. Thus, from now on, we assume that Θ is a pure inner function.

Recall that the defect operators and spaces of a contraction T are defined by $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T = \text{ran } D_T$. The next lemma shows how one can identify the defect spaces of S_Θ ; a good reference is [7, Section 1].

Lemma 2.2. *Suppose $\Theta(z) : \mathcal{E} \rightarrow \mathcal{E}_*$ is a pure inner function; in particular, $D_{\Theta(0)}$ and $D_{\Theta(0)^*}$ have dense ranges. Define the maps $\iota : \mathcal{E} \rightarrow H^2(\mathcal{E}_*)$, $\iota_* : \mathcal{E}_* \rightarrow H^2(\mathcal{E}_*)$ (on dense domains) by*

$$\begin{aligned} \iota(D_{\Theta(0)}\xi) &= \frac{1}{z}(\Theta(z) - \Theta(0))\xi, \quad \xi \in \mathcal{E}; \\ \iota_*(D_{\Theta(0)^*}\xi_*) &= (I - \Theta(z)\Theta(0)^*)\xi_*, \quad \xi_* \in \mathcal{E}_*. \end{aligned} \quad (2.3)$$

Then ι and ι_* are isometries with ranges \mathcal{D}_{S_Θ} and $\mathcal{D}_{S_\Theta^*}$ respectively, and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\iota} & \mathcal{D}_{S_\Theta} \\ \downarrow -\Theta(0) & & \downarrow S_\Theta \\ \mathcal{E}_* & \xrightarrow{\iota_*} & \mathcal{D}_{S_\Theta^*} \end{array} \quad (2.4)$$

In particular, $\dim \mathcal{D}_{S_\Theta} = \dim \mathcal{E}$ and $\dim \mathcal{D}_{S_\Theta^*} = \dim \mathcal{E}_*$.

We will occasionally write ι^Θ and ι_*^Θ to indicate the dependence on Θ .

From the Sz-Nagy–Foias theory it follows that any C_0 contraction T (that is, a contraction such that the powers of the adjoint tend strongly to 0) is unitarily equivalent to some S_Θ , where we can take $\mathcal{E} = \mathcal{D}_T$ and $\mathcal{E}_* = \mathcal{D}_{T^*}$.

We are actually interested in the particular case when $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$. The following definition appears in [19].

Definition 2.3. A contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of class $C_0(N)$ if $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$ and T^n, T^{*n} tend strongly to 0.

If $T \in C_0(N)$, then T is unitarily equivalent to S_Θ , with $\Theta(z) : \mathbb{C}^N \rightarrow \mathbb{C}^N$. In this case $\Theta(0)$ is a strict contraction, and formulas (2.3) are defined on all of \mathbb{C}^N . The next lemma collects a few facts that we shall use.

Lemma 2.4. Suppose $\Theta(z) : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is an inner function.

- (i) K_Θ is finite dimensional if and only if Θ is a finite Blaschke–Potapov product.
- (ii) If $\Theta_n \rightarrow \Theta$ in the uniform norm, then $P_{K_{\Theta_n}} \rightarrow P_{K_\Theta}$ uniformly.
- (iii) Let $b_j = b(P_j, \lambda_j)$. If $B = \prod_k b_k$ is an infinite Blaschke–Potapov product and $B_n = b_1 \cdots b_n$, then
 - (a) $K_B = \bigcup_n K_{B_n}$;
 - (b) $B_n \xi \rightarrow B\xi$ in $H^2(\mathbb{C}^N)$, for any $\xi \in \mathbb{C}^N$.

Proof. Statement (i) can be found, for instance, in [18, Ch.2, Lemma 5.1], while (ii) follows from (2.2).

As for (iii), a standard normal family argument shows that $BH^2(\mathbb{C}^N) = \bigcap_n B_n H^2(\mathbb{C}^N)$, and therefore (a) follows by passing to orthogonal complements.

For (b), write $B = B_n \tilde{B}_n$, where \tilde{B}_n is also an infinite Blaschke–Potapov product. If $B(0)$ is invertible, the pointwise convergence of B_n to B implies that $\tilde{B}_n(0) \rightarrow I_{\mathbb{C}^N}$, whence (taking norms and scalar products in $H^2(\mathbb{C}^N)$)

$$\begin{aligned} \|B_n \xi - B\xi\|^2 &= 2\|\xi\|^2 - 2\Re\langle B_n \xi, B\xi \rangle = 2\|\xi\|^2 - 2\Re\langle \xi, \tilde{B}_n \xi \rangle \\ &= 2\|\xi\|^2 - 2\Re\langle \xi, \tilde{B}_n(0)\xi \rangle \rightarrow 0. \end{aligned}$$

In the general case, write $B_n = CD_n$, where C contains the Blaschke–Potapov factors $b(P, \lambda)$ corresponding to $\lambda = 0$. We have then $B = CD$ (with D an infinite Blaschke–Potapov product), while the previous argument shows that

$D_n \xi \rightarrow D\xi$ in $H^2(\mathbb{C}^N)$. Multiplying with the inner function C yields the result. \square

3. Unitary N -dilations

The next result is folklore; we give a short proof.

Proposition 3.1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$. If $U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{E})$ is a unitary N -dilation of T , then there exist unitary operators $\omega : \mathcal{E} \rightarrow \mathcal{D}_T, \omega_* : \mathcal{E} \rightarrow \mathcal{D}_{T^*}$, such that*

$$U = \begin{pmatrix} T & D_{T^*}\omega \\ \omega_*^* D_T & -\omega_*^* T^* \omega \end{pmatrix}. \quad (3.1)$$

Conversely, any choice of $\omega : \mathcal{E} \rightarrow \mathcal{D}_T, \omega_ : \mathcal{E} \rightarrow \mathcal{D}_{T^*}$ yields, through formula (3.1), a unitary N -dilation of T .*

Proof. Theorem 1.3 of [1] says that if

$$\begin{pmatrix} T & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \mathcal{H} \oplus \mathcal{E} \rightarrow \mathcal{H} \oplus \mathcal{E}$$

is a contraction, then there exist contractions $\Gamma_1 : \mathcal{E} \rightarrow \mathcal{D}_{T^*}, \Gamma_2 : \mathcal{D}_T \rightarrow \mathcal{E}$ and $\Gamma : \mathcal{D}_{\Gamma_1} \rightarrow \mathcal{D}_{\Gamma_2^*}$ such that $T_{12} = D_{T^*}\Gamma_1, T_{21} = \Gamma_2 D_T$ and $T_{22} = -\Gamma_2 T^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1}$. We apply this result to U .

Since

$$\|U(x \oplus 0)\|^2 = \|Tx\|^2 + \|\Gamma_2 D_T x\|^2 \leq \|Tx\|^2 + \|D_T x\|^2 = \|x\|^2,$$

and the first column of U is an isometry, the last term is equal to the first; so the middle inequality is an equality. This means that Γ_2 acts isometrically on the image of D_T ; but this is precisely \mathcal{D}_T , whence Γ_2 has to be an isometry.

In fact, Γ_2 is unitary, since it acts between spaces of the same dimension N . Similarly, we obtain that Γ_1 is unitary, which implies Γ acts between 0 spaces. The result follows if we let $\omega = \Gamma_1$ and $\omega_* = \Gamma_2^*$.

The converse is immediate. \square

We can write (3.1) as

$$U = \begin{pmatrix} I & 0 \\ 0 & \omega_*^* \end{pmatrix} \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}.$$

The next corollary follows immediately from this formula.

Corollary 3.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction with $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$ and*

$$\mathbf{u} = \begin{pmatrix} T & \mathbf{u}_{12} \\ \mathbf{u}_{21} & \mathbf{u}_{22} \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{E})$$

is a unitary N -dilation of T . Then any unitary N -dilation U of T on $\mathcal{H} \oplus \mathcal{E}'$ (for some Hilbert space \mathcal{E}' with $\dim \mathcal{E}' = N$) is given by the formula

$$U = \begin{pmatrix} I & 0 \\ 0 & \Omega_*^* \end{pmatrix} \mathbf{u} \begin{pmatrix} I & 0 \\ 0 & \Omega \end{pmatrix} \quad (3.2)$$

where $\Omega, \Omega_* : \mathcal{E}' \rightarrow \mathcal{E}$ are unitaries.

We are interested in consequences for S_Θ . The notation below refers to that of Lemma 2.2.

Lemma 3.3. *All unitary N -dilations of S_Θ to $K_\Theta \oplus \mathbb{C}^N$ can be indexed by unitaries $\Omega, \Omega_* : \mathbb{C}^N \rightarrow \mathbb{C}^N$, according to the formula*

$$U_{\Omega, \Omega_*} = \begin{pmatrix} S_\Theta & \iota_* D_{\Theta(0)^*} \Omega \\ \Omega_*^* D_{\Theta(0)} \iota^* & \Omega_*^* \Theta(0)^* \Omega \end{pmatrix}. \quad (3.3)$$

Proof. Let us apply Proposition 3.1 (the part stated as a converse) to the case $T = S_\Theta$, $\mathcal{E} = \mathbb{C}^N$, $\omega = \iota_*$, $\omega_* = \iota$. We obtain the following N -dilation of T to $\mathcal{H} \oplus \mathbb{C}^N$:

$$V = \begin{pmatrix} S_\Theta & D_{S_\Theta^*} \iota_* \\ \iota^* D_{S_\Theta} & -\iota^* S_\Theta^* \iota_* \end{pmatrix}.$$

The commutative diagram (2.4) yields the relations

$$S_\Theta \iota = -\iota_* \Theta(0), \quad \iota^* S_\Theta^* = -\Theta(0)^* \iota_*^*. \quad (3.4)$$

It follows immediately that $-\iota^* S_\Theta^* \iota_* = \Theta(0)^*$.

From (3.4) we have $\iota^* S_\Theta^* S_\Theta \iota = \Theta(0)^* \Theta(0)$, whence $\iota^* (I_{\mathcal{H}} - S_\Theta^* S_\Theta) \iota = I_{\mathbb{C}^N} - \Theta(0)^* \Theta(0)$ and thus $\iota^* D_{S_\Theta} \iota = D_{\Theta(0)}$. Multiplying the last relation with ι^* on the right, and taking into account that $\iota^* = P_{\mathcal{D}_{S_\Theta}}$ and $D_{S_\Theta} P_{\mathcal{D}_{S_\Theta}} = D_{S_\Theta}$, we obtain

$$\iota^* D_{S_\Theta} = D_{\Theta(0)} \iota^*.$$

A similar computation yields

$$D_{S_\Theta^*} \iota_* = \iota_* D_{\Theta(0)^*},$$

and thus

$$V = \begin{pmatrix} S_\Theta & \iota_* D_{\Theta(0)^*} \\ D_{\Theta(0)} \iota^* & \Theta(0)^* \end{pmatrix}.$$

Applying Corollary 3.2 to $\mathcal{E} = \mathcal{E}' = \mathbb{C}^N$ and $\mathbf{u} = V$ finishes the proof. \square

We have thus parametrized all unitary N -dilations of S_Θ to $K_\Theta \oplus \mathbb{C}^N$ by pairs of unitaries on \mathbb{C}^N . If we are interested only in classes of unitary equivalence, we may take a single unitary as parameter, since U_{Ω, Ω_*} , $U_{\Omega \Omega_*^*, I}$, and $U_{I, \Omega_* \Omega^*}$ are all unitarily equivalent, and therefore have the same numerical range. In the sequel, we let

$$U_\Omega^\Theta := \begin{pmatrix} S_\Theta & \iota_* D_{\Theta(0)^*} \Omega \\ D_{\Theta(0)} \iota^* & \Theta(0)^* \Omega \end{pmatrix}. \quad (3.5)$$

4. The main result

Let \mathfrak{K} denote the complete metric space of all nonempty compact subsets of \mathbb{C} , endowed with the Hausdorff distance \mathfrak{d} . Suppose that $A \in \mathfrak{K}$ and $\tau : X \rightarrow \mathfrak{K}$ is a continuous mapping defined on some compact space X . We will say that τ *wraps* A if for each open half-plane \mathbb{H} in \mathbb{C} that contains A there exists $x \in X$ such that $\tau(x) \subset \mathbb{H}$.

Lemma 4.1. *Let $A_n, A \in \mathfrak{K}$ with $A_n \rightarrow A$. Let X be a compact space, and $\tau_n, \tau : X \rightarrow \mathfrak{K}$ be continuous mappings such that $\tau_n \rightarrow \tau$ uniformly on X . Suppose that for each n , τ_n wraps A_n . Then τ wraps A .*

Proof. If \mathbb{H} is an open half-plane and $A \subset \mathbb{H}$, let \mathbb{H}' be a slight translate of \mathbb{H} towards A such that we still have $A \subset \mathbb{H}'$. For n sufficiently large $A_n \subset \mathbb{H}'$. It follows then from the assumption that for each n sufficiently large there exists $x_n \in X$ such that $\tau_n(x_n) \subset \mathbb{H}'$. Letting x be a limit point of x_n in X , a simple $\epsilon/2$ argument shows that $\tau(x) \subset \mathbb{H}$. \square

Remark 4.2. Suppose $A \subset \tau(x)$ for all x , and A and $\tau(x)$ are convex for all x . If τ wraps A , then $A = \bigcap_{x \in X} \tau(x)$. The converse is not true, as can easily be seen by considering A to be the intersection of two line segments. However, the result that we quote below (in Theorem 4.6, Step 1) from [8] actually yields a wrapping property of A , not only intersection.

The following simple lemma will be used in Section 6.

Lemma 4.3. *Suppose $A \in \mathfrak{K}$ and $\tau : X \rightarrow \mathfrak{K}$ wraps A . If $B \in \mathfrak{K}$, $\tilde{A} := \text{co}(A, B)$, $\tilde{\tau}(x) = \text{co}(\tau(x), B)$, then $\tilde{\tau}$ wraps \tilde{A} .*

Proof. Take a half-plane \mathbb{H} that contains \tilde{A} . Then it contains A and B . By hypothesis, there exists $x \in X$ such that $\tau(x) \subset \mathbb{H}$. Since \mathbb{H} is convex, it follows that $\tilde{\tau}(x) \subset \mathbb{H}$, which proves the lemma. \square

The elements of \mathfrak{K} that we will consider are closures of numerical ranges. The next lemma states some continuity properties for these sets.

Lemma 4.4. (i) *Let $T, S \in \mathcal{L}(H)$. Then $\mathfrak{d}(\overline{W(T)}, \overline{W(S)}) \leq \|T - S\|$.*

(ii) *If $H_n \subset H_{n+1} \subset \dots \subset H$ and $\bigcup_n H_n = H$, then for all $T \in \mathcal{L}(H)$,*

$$\overline{W(T)} = \bigcup_n \overline{W(P_{H_n} T P_{H_n} | H_n)}.$$

In particular, $\mathfrak{d}(\overline{W(P_{H_n} T P_{H_n} | H_n)}, \overline{W(T)}) \rightarrow 0$.

(iii) *Suppose $T \in \mathcal{L}(H)$, and P, Q are orthogonal projections on H , with $\|P - Q\| < 1$. Then*

$$\mathfrak{d}(\overline{W(PTP|PH)}, \overline{W(QTQ|QH)}) \leq \|T\| \cdot \|P - Q\| \left[1 + \frac{2}{(1 - \|P - Q\|)^2} \right].$$

In particular, if P_n, P are orthogonal projections and $P_n \rightarrow P$ uniformly, then $\mathfrak{d}(\overline{W(P_n T P_n | P_n H)}, \overline{W(PTP|PH)}) \rightarrow 0$.

In the sequel we will let X denote the space of unitary operators on \mathbb{C}^N and we define $\tau^\Theta(\Omega) = \overline{W(U_\Omega^\Theta)}$, where U_Ω^Θ is given by (3.5).

The next lemma singles out a technical argument that will be used twice in the proof of Theorem 4.6.

Lemma 4.5. *Suppose that $\Theta, \Theta_n : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$ are inner functions, such that*

- (a) $\Theta_n \xi \rightarrow \Theta \xi$ in $H^2(\mathbb{C}^N)$, for any $\xi \in \mathbb{C}^N$;
- (b) $\mathfrak{d}(\overline{W(S_{\Theta_n})}, \overline{W(S_\Theta)}) \rightarrow 0$;

(c) if we define

$$V_{n,\Omega} = \begin{pmatrix} S_{\Theta_n} & P_{K_{\Theta_n}} \iota_* D_{\Theta(0)^*} \Omega \\ D_{\Theta(0)} \iota^* P_{K_{\Theta_n}} & \Theta(0)^* \Omega \end{pmatrix},$$

then $\mathfrak{d}(\overline{W(V_{n,\Omega})}, \overline{W(U_\Omega^\Theta)}) \rightarrow 0$ uniformly in Ω .

If τ^{Θ_n} wraps $\overline{W(S_{\Theta_n})}$ for all n , then τ^Θ wraps $\overline{W(S_\Theta)}$.

Proof. Condition (a) implies, by formulas (2.3), that $\iota^{\Theta_n} \rightarrow \iota^\Theta$ and $\iota_*^{\Theta_n} \rightarrow \iota_*^\Theta$; whence $\|V_{n,\Omega} - U_\Omega^\Theta\| \rightarrow 0$ uniformly in Ω . Therefore $\mathfrak{d}(\overline{W(V_{n,\Omega})}, \overline{W(U_\Omega^\Theta)}) \rightarrow 0$ uniformly in Ω , which, together with (c), yields $\mathfrak{d}(\overline{W(U_\Omega^\Theta)}, \overline{W(U_\Omega^\Theta)}) \rightarrow 0$ uniformly in Ω .

We may then apply Lemma 4.1 with $A_n = \overline{W(S_{\Theta_n})}$, $A = \overline{W(S_\Theta)}$, $\tau_n = \tau^{\Theta_n}$, and $\tau = \tau^\Theta$. With these notations, the assumption of Lemma 4.5 becomes that τ_n wraps A_n , and it follows that τ wraps A . \square

Theorem 4.6. *For any inner function $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$, the map τ^Θ wraps $\overline{W(S_\Theta)}$.*

Proof. The proof will be done in three steps.

Step 1. In case Θ is a finite Blaschke–Potapov product, the space K_Θ is finite dimensional and the statement is a consequence of [8, Theorem 1.2] (see Remark 4.2).

Step 2. To pass to infinite Blaschke–Potapov products, suppose that $\Theta = B$ and $\Theta_n = B_n$ (where the notation is as in Lemma 2.4 (iii)). We want to use Lemma 4.5. Condition (a) therein is satisfied by Lemma 2.4 (iii)(b). Applying Lemma 4.4 (ii) to $T = S_B$, $H = K_B$, $H_n = K_{B_n}$, we obtain $\mathfrak{d}(\overline{W(S_{\Theta_n})}, \overline{W(S_\Theta)}) \rightarrow 0$, and therefore (b) is also satisfied. Finally, to obtain (c), we apply Lemma 4.4 (ii) again, this time to $T = U_\Omega^B$, $H = K_B \oplus \mathbb{C}^N$, $H_n = K_{B_n} \oplus \mathbb{C}^N$. By Step 1 we know that τ^{B_n} wraps $\overline{W(S_{B_n})}$ for all n , and Lemma 4.5 implies that τ^B wraps $\overline{W(S_B)}$.

Step 3. According to Lemma 2.1, we take a sequence of Blaschke–Potapov products Θ_n that tend uniformly to an arbitrary inner function Θ . Condition (a) in Lemma 4.5 is obviously satisfied. By Lemma 2.4 (ii), we have $P_{K_{\Theta_n}} \rightarrow P_{K_\Theta}$ uniformly. Since $S_{\Theta_n} = P_{K_{\Theta_n}} \mathbf{T}_z P_{K_{\Theta_n}}|_{K_{\Theta_n}}$ and $S_\Theta = P_{K_\Theta} \mathbf{T}_z P_{K_\Theta}|_{K_\Theta}$, Lemma 4.4 (iii), applied to $H = H^2(\mathbb{C}^N)$, $T = \mathbf{T}_z$, $P_n = P_{K_{\Theta_n}}$, and $P = P_{K_\Theta}$, yields condition (b) in Lemma 4.5.

To obtain (c), apply Lemma 4.4 (iii) again, this time to $H = H^2(\mathbb{C}^N) \oplus \mathbb{C}^N$, $P_n = P_{K_{\Theta_n} \oplus \mathbb{C}^N}$, $P = P_{K_\Theta \oplus \mathbb{C}^N}$, and

$$T = \begin{pmatrix} T_z & \iota_* D_{\Theta(0)^*} \Omega \\ D_{\Theta(0)} \iota^* & \Theta(0)^* \Omega \end{pmatrix}.$$

Once again, we use the fact that $P_{K_{\Theta_n}} \rightarrow P_{K_\Theta}$ uniformly to conclude that (c) is also satisfied.

By Step 2 we know that τ^{Θ_n} wraps $\overline{W(S_{\Theta_n})}$ for all n , and Lemma 4.5 implies that τ^Θ wraps $\overline{W(S_\Theta)}$. The proof of the theorem is finished. \square

The next corollary is a consequence of Remark 4.2.

Corollary 4.7. *Suppose $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$ is an inner function. Then*

$$\overline{W(S_\Theta)} = \bigcap_{\Omega} \overline{W(U_\Omega^\Theta)}, \quad (4.1)$$

where U_Ω^Θ is defined by (3.5), while the intersection is taken with respect to all unitary operators Ω on \mathbb{C}^N .

Since $C_0(N)$ contractions are unitarily equivalent to model operators S_Θ , with $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$ inner, we may extend the result to this class.

Theorem 4.8. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction of class $C_0(N)$, \mathcal{U} is the set of unitary N -dilations of T to $\mathcal{H} \oplus \mathbb{C}^N$, and $\tau : \mathcal{U} \rightarrow \mathfrak{K}$ is defined by $\tau(\mathbf{U}) = \overline{W(\mathbf{U})}$. Then τ wraps $\overline{W(T)}$. In particular,*

$$\overline{W(T)} = \bigcap_{\mathbf{U} \in \mathcal{U}} \overline{W(\mathbf{U})}.$$

5. Spectrum and numerical range of N -dilations

In the case that Θ is a finite (scalar) Blaschke product, the spectrum of the extensions U_Ω^Θ can be identified precisely. Since U_Ω^Θ is a unitary operator, the numerical range is the (closed) convex hull of the spectrum, and we obtain a complete description of $W(U_\Omega^\Theta)$, (see, for example, [3], [5], and [9]). The same can be done in the case of a general matrix-valued inner function, by relating these functions to perturbations of a “slightly larger” model operator. We need some preliminary material, for which the reference is [7].

If $T \in \mathcal{L}(\mathcal{H})$ is a contraction, then $T(\mathcal{D}_T) \subset \mathcal{D}_{T^*}$, $T(\mathcal{D}_T^\perp) \subset \mathcal{D}_{T^*}^\perp$, and T acts unitarily from \mathcal{D}_T^\perp onto $\mathcal{D}_{T^*}^\perp$. For $A : \mathcal{D}_T \rightarrow \mathcal{D}_{T^*}$, define $T[A] \in \mathcal{L}(\mathcal{H})$ by the formula

$$T[A]x = \begin{cases} Ax & \text{if } x \in \mathcal{D}_T, \\ Tx & \text{if } x \in \mathcal{D}_T^\perp. \end{cases} \quad (5.1)$$

It is easy to see that $T[A]$ is a contraction (respectively isometry, coisometry, unitary) if and only if A is a contraction (respectively isometry, coisometry, unitary).

We will be interested in the particular situation when $T = S_\Xi$, with $\Xi(z) = z\Theta(z)$, with $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$ an inner function and A unitary. According to formulas (2.3), we have then $\mathcal{D}_{S_\Xi} = \Theta\mathbb{C}^N$, $\mathcal{D}_{S_\Xi^*} = \mathbb{C}^N$. Thus a unitary mapping $A : \mathcal{D}_T \rightarrow \mathcal{D}_{T^*}$ is given by $A\Theta(z)\xi = \omega\xi$, with $\omega : \mathbb{C}^N \rightarrow \mathbb{C}^N$ unitary. We will write $A = A_\omega$. Since

$$K_\Xi = K_\Theta \oplus \Theta\mathbb{C}^N = zK_\Theta \oplus \mathbb{C}^N,$$

we have unitary operators $J, J_* : K_\Theta \oplus \mathbb{C}^N \rightarrow K_\Xi$ defined by

$$J(f \oplus \xi) = f + \Theta\xi, \quad J_*(f \oplus \xi) = zf + \xi.$$

We will write

$$Z_\Xi(\omega) := J^* S_\Xi[A_\omega] J \in \mathcal{L}(K_\Theta \oplus \mathbb{C}^N).$$

With these notations, the lemma below follows from [7, Theorem 3.6]. For a more general result, see [2, Theorem 4.5].

Lemma 5.1. *With the above assumptions, the spectrum of $Z_\Xi(\omega)$ is the union of the sets of points $\zeta \in \mathbb{T}$ at which Ξ has no analytic continuation and the set of points $\zeta \in \mathbb{T}$ at which Ξ has an analytic continuation but $\Xi(\zeta) - \omega$ is not invertible.*

In particular, if Ξ is a finite Blaschke–Potapov product, then

$$\sigma(Z_\Xi(\omega)) = \{\zeta \in \mathbb{T} : \det(\Xi(\zeta) - \omega) = 0\}.$$

The relation with N -dilations is given by the next proposition.

Proposition 5.2. *Suppose $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}^N)$ is an inner function. Define $\Xi(z) = z\Theta(z)$. Then $U_\Omega^\Theta = Z_\Xi(\Omega)$.*

Proof. We have

$$Z_\Xi(\Omega) = J^* S_\Xi[A_\Omega] J = (J^* J_*)(J_*^* S_\Xi[A_\Omega] J). \quad (5.2)$$

Since

$$S_\Xi[A_\Omega] J(f \oplus \xi) = S_\Xi[A_\Omega](f + \Theta\xi) = zf + \Omega\xi = J_*(f \oplus \Omega\xi),$$

it follows that

$$(J_*^* S_\Xi[A_\Omega] J)(f \oplus \xi) = f \oplus \Omega\xi. \quad (5.3)$$

To compute $J^* J_* : K_\Theta \oplus \mathbb{C}^N \rightarrow K_\Theta \oplus \mathbb{C}^N$, denote the corresponding matrix by $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, and let P_i denote the projection on the i -th component in $K_\Theta \oplus \mathbb{C}^N$. We have

$$A_{11}(f) = P_1(J^* J_*(f \oplus 0)) = P_1(J^* zf) = P_{K_\Theta} zf = S_\Theta f.$$

Further, $J_*(0 \oplus \xi) = \xi$, viewed as a constant function in K_Ξ . This decomposes with respect to $K_\Xi = K_\Theta \oplus \Theta\mathbb{C}^N$ as

$$\xi = (1 - \Theta\Theta(0)^*)\xi + \Theta\Theta(0)^*\xi = \iota_* D_{\Theta(0)^*} \xi + \Theta\Theta(0)^*\xi.$$

It follows that

$$J^* J_*(0 \oplus \xi) = \iota_* D_{\Theta(0)^*} \xi \oplus \Theta(0)^*\xi,$$

and thus

$$A_{12} = \iota_* D_{\Theta(0)^*}, \quad A_{22} = \Theta(0)^*.$$

To obtain A_{21} , we work now with the adjoint map $J_*^* J$. We have $J(0 \oplus \xi) = \Theta\xi$, and the last function decomposes with respect to $K_\Xi = zK_\Theta \oplus \mathbb{C}^N$ as

$$\Theta\xi = z \left(\frac{\Theta - \Theta(0)}{z} \right) \xi + \Theta(0)\xi = z\iota D_{\Theta(0)} \xi + \Theta(0)\xi,$$

whence

$$J_*^* J(0 \oplus \xi) = \iota D_{\Theta(0)} \xi \oplus \Theta(0)\xi.$$

Therefore

$$A_{21}^* = \iota D_{\Theta(0)}, \quad A_{21} = D_{\Theta(0)} \iota^*.$$

Finally,

$$J^* J = \begin{pmatrix} S_\Theta & \iota_* D_{\Theta(0)^*} \\ D_{\Theta(0)} \iota^* & \Theta(0)^* \end{pmatrix} \quad (5.4)$$

Now the proof follows by comparing equations (5.2), (5.3), and (5.4) with (3.5). \square

From Lemma 5.1 and Proposition 5.2, the final result about spectrum and numerical range of N -dilations follows.

Theorem 5.3. *With the above notations, the spectrum $\sigma(U_\Omega^\Theta)$ is the union of the sets of points $\zeta \in \mathbb{T}$ at which Θ has no analytic continuation and the set of points $\zeta \in \mathbb{T}$ at which Θ has an analytic continuation but $\zeta\Theta(\zeta) - \Omega$ is not invertible, while $\overline{W(U_\Omega^\Theta)}$ is the closed convex hull of $\sigma(U_\Omega^\Theta)$.*

In particular, if Θ is a finite Blaschke–Potapov product, then

$$\sigma(U_\Omega^\Theta) = \{\zeta \in \mathbb{T} : \det(\zeta\Theta(\zeta) - \Omega) = 0\}$$

and $\overline{W(U_\Omega^\Theta)}$ is the closed convex hull of the zeros of the polynomial $\det(\zeta\Theta(\zeta) - \Omega)$.

The scalar case of Theorem 5.3 is contained in [11, Theorem 6.3].

6. Final remarks

It seems natural to formulate the following conjecture, which would complement Choi and Li’s answer to Halmos’ question.

Conjecture 6.1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction with $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$, \mathcal{U} is the set of unitary N -dilations of T to $\mathcal{H} \oplus \mathbb{C}^N$, and $\tau : \mathcal{U} \rightarrow \mathfrak{K}$ is defined by $\tau(\mathbf{U}) = \overline{W(\mathbf{U})}$. Then τ wraps $\overline{W(T)}$. In particular,*

$$\overline{W(T)} = \bigcap_{\mathbf{U} \in \mathcal{U}} \overline{W(\mathbf{U})} \quad (6.1)$$

Note that the conjecture is open even for $N = 1$. The main points that have been settled are presented below. In the sequel $T \in \mathcal{L}(\mathcal{H})$ will be a contraction with $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$.

6.1. Theorem 4.8 shows that the conjecture is true for $C_0(N)$ contractions.

6.2. As we show below, if we add a unitary operator to one for which the conjecture holds, the conjecture will still hold.

Lemma 6.2. *If $T_1 = T \oplus V$, where V is unitary, and Conjecture 6.1 is true for T , then it is true for T_1 .*

Proof. The unitary N -dilations of $T \oplus V$, for V unitary, are exactly $\mathbf{U} \oplus V$, with \mathbf{U} a unitary N -dilation of T . Since the numerical range of a direct sum is the convex hull of the numerical ranges of the components, the statement follows from Lemma 4.3. \square

Again by [19], it is known that an arbitrary contraction is the direct sum of a completely nonunitary contraction and a unitary; it follows then from Lemma 6.2 that it is enough to prove Conjecture 6.1 for a completely nonunitary T .

6.3. We now specialize to the case $N = 1$. Suppose T is a completely nonunitary contraction with scalar characteristic function θ [19]; now θ is an arbitrary function in the unit ball of H^∞ . Then T is unitarily equivalent to the model operator $\mathbf{T}_\theta \in \mathcal{L}(\mathbf{K}_\theta)$, where

$$\mathbf{K}_\theta = (H^2 \oplus L^2(\Delta)) \ominus \{\theta f \oplus (1 - |\theta|^2)^{1/2} f : f \in H^2\},$$

with $\Delta = \{\zeta \in \mathbb{T} : |\theta(\zeta)| < 1\}$, while $\mathbf{T}_\theta(f \oplus g) = P_{\mathbf{K}_\theta}(zf \oplus \zeta g)$.

If θ is inner, then $\Delta = \emptyset$ and we are back in the $C_0(1)$ case discussed in 6.1.

On the other hand, the spectrum of \mathbf{T}_θ may be precisely identified in terms of the characteristic function: $\sigma(\mathbf{T}_\theta)$ is the union of the zeros of θ inside \mathbb{D} and the complement of the open arcs of \mathbb{T} on which $|\theta(\zeta)| = 1$ and through which θ has an analytic extension outside the unit disk (see again [19] for a general statement; in the scalar case it was known earlier and is usually called the Livsic–Moeller theorem).

In particular, it follows that Conjecture 6.1 can be settled for a situation at the opposite extreme of the case in which θ is inner. Namely, if $|\theta(\zeta)| < 1$ almost everywhere on \mathbb{T} , then $\sigma(\mathbf{T}_\theta) \supset \mathbb{T}$. In this case, Conjecture 6.1 is trivially true: $\overline{W(T)}$ as well as every $\overline{W(U)}$ must equal $\overline{\mathbb{D}}$.

A final remark: the case $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$ is the only one in which we can hope to obtain the numerical range of T by using “economical” unitary dilations. If $\dim \mathcal{D}_T \neq \dim \mathcal{D}_{T^*}$, or if both dimensions are infinite, then it is easy to see that for any unitary dilation $U \in \mathcal{L}(\mathcal{K})$ of $T \in \mathcal{L}(\mathcal{H})$ one must have $\dim(\mathcal{K} \ominus \mathcal{H}) = \infty$.

Acknowledgements

We thank the referee for useful remarks and, in particular, for pointing out the need for a slight change to the proof of Theorem 4.6.

References

- [1] Arsene, Gr., Gheondea, A., Completing matrix contractions, *J. Operator Theory* **7** (1982), 179–189.
- [2] Ball, J.A., Lubin, A., On a class of contractive perturbations of restricted shifts, *Pacific J. Math.* **63** (1976), 309–323.
- [3] Chalendar, I., Gorkin, P., Partington, J. R., Numerical ranges of restricted shifts and unitary dilations, *Oper. Matrices* **3** (2009), 271–281.
- [4] Choi M.-D., Li C.-K., Constrained unitary dilations and numerical ranges, *J. Operator Theory* **46** (2001), 435–447.

- [5] Daepf, U., Gorkin, P., Voss, K., Poncelet's Theorem, Sendov's Conjecture and Blaschke Products, *J. Math. Anal. App.* **365** (2010), 93-102.
- [6] Durszt, E., On the numerical range of normal operators, *Acta Sci. Math. (Szeged)* **25** (1964), 262-265.
- [7] Fuhrmann, P.A., On a class of finite dimensional contractive perturbations of restricted shifts of finite multiplicity, *Israel J., Math.* **16** (1973), 162-175.
- [8] Gau, H.-L., Li, C.-K, Wu, P.Y., Higher-rank numerical ranges and dilations, *J. Operator Theory* **63** (2010), 181-189.
- [9] Gau, H.-L., Wu, P. Y., Numerical range of $S(\phi)$, *Linear and Multilinear Algebra* **45** (1998), no. 1, 49-73.
- [10] Gau, H.-L., Wu, P. Y., Numerical range circumscribed by two polygons, *Linear Algebra Appl.* **382** (2004), 155-170.
- [11] Gau, H.-L., Wu, P. Y., Numerical range and Poncelet property, *Taiwanese J. Math.* **7** (2003), no. 2, 173-193.
- [12] Gau, H.-L., Wu, P. Y., Dilation to inflations of $S(\phi)$, *Linear and Multilinear Algebra* **45** (1998), no. 2-3, 109-123.
- [13] Halmos, P.R., Numerical ranges and normal dilations, *Acta Sci. Math. (Szeged)* **25** (1964), 1-5.
- [14] Halmos, P. R., *A Hilbert space problem book*. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. Springer-Verlag, New York-Berlin, 1982.
- [15] Katsnelson, V. E., Kirstein, B., On the theory of matrix-valued functions belonging to the Smirnov class. *Topics in interpolation theory* 299-350, Oper. Theory Adv. Appl., 95, Birkhäuser, Basel, 1997.
- [16] Mirman, B., Borovikov, V., Ladyzhensky, L., Vinograd, R., Numerical ranges, Poncelet curves, invariant measures, *Linear Algebra Appl.* **329** (2001), no. 1-3, 61-75.
- [17] Mirman, B., Numerical ranges and Poncelet curves, *Linear Algebra Appl.* **281** (1998), no. 1-3, 59-85.
- [18] Peller, V.V., *Hankel Operators and their Applications*, Springer Verlag, 2003.
- [19] Sz-Nagy, B., Foias, C., *Harmonic Analysis of Operators on Hilbert Space*, North-Holland Publishing, 1970.
- [20] Wu, P.Y., Polygons and numerical ranges, *Amer. Math. Monthly*, **107** (2000), no. 6, 528-540.

Chafiq Benhida

UFR de Mathématiques, Université des Sciences et Technologies de Lille, F-59655 Villeneuve D'Ascq Cedex, France

e-mail: Chafiq.Benhida@math.univ-lille1.fr

Pamela Gorkin

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.

e-mail: pgorkin@bucknell.edu

Dan Timotin

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania

e-mail: Dan.Timotin@imar.ro